

INTERACTIVE MULTIOBJECTIVE TABU/SCATTER  
SEARCH BASED ON REFERENCE POINT

BÚSQUEDA TABÚ/DISPERSA MULTIOBJETIVO  
INTERACTIVAS BASADAS EN PUNTO DE  
REFERENCIA

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### Abstract

This paper presents multiobjective tabu/scatter search architecture with preference information based on reference points for problems of continuous nature. Features of this new version are: its interactive behavior, its deterministic approximation to Pareto-optimality solutions near the reference point, and the possibility to change progressively the reference point to explore different preference regions. The approach does not impose any restrictions with respect to the location of the reference points in the objective space. On 2-objective to 10-objective optimization test problems the modified approach shows its efficacy and efficiency to find an adequate non-dominated set of solutions in the preferred region.

**Keywords:** multiple objectives; metaheuristics; reference point; continuous optimization.

### Resumen

Este artículo presenta una arquitectura Tabú/Búsqueda Dispersa multiobjetivo, con información de preferencia basada en punto de referencia para problemas de naturaleza continua. Los rasgos de esta nueva versión son los siguientes: funcionamiento interactivo, aproximación determinística a las soluciones Pareto cercanas al punto de referencia y la posibilidad de cambiar el punto de referencia para explorar diferentes regiones de preferencia. El enfoque no impone restricciones con relación a los puntos de referencia en el espacio de los objetivos, y muestra su habilidad en la solución de problemas desde 2 hasta más de 10 objetivos, hallando conjuntos de soluciones eficientes cercanas al punto de referencia.

**Palabras clave:** múltiples objetivos; metaheurísticas; punto de referencia; optimización continua.

**Mathematics Subject Classification:** 90C27.

## 1 Introduction

In the last decade the multiobjective metaheuristics (MOMHs) have been successfully used to deal with many different application problems. Nevertheless, these approaches have difficulties dealing with more than 5 objectives. Generation of a high quality approximation to the Pareto front demands too computational time and in the majority of the cases the developed approaches do not perform well. The general observation is that the effectiveness of the MOMHs decline with the increase of the number of the objectives. An important task to the decision-maker (DM) is to choose a single preferred solution. In the last

time special attention is shown in the development of methods utilizing preference information. In Branke et al. [5], user preferences were taken into account by modifying the definition of dominance. Fonseca and Fleming [12] proposed using MOGA together with goal information as an additional criterion to assign ranks to the members of a population. Cvetkovic and Parmee [6] used binary preference relations translated into weights, to narrow the search modifying the concept of dominance. Barbosa and Barreto [2] proposed a co-evolutionary genetic algorithm with two populations: a population of solutions to the problem and a population of weights. Deb [7] modified his NSGA approach to find a set of solutions which are closest to the supplied goal point. Deb et al. [11] included preferences through the use of reference points and guided dominance scheme and a biased crowding scheme was suggested. Thiele et al. [17] incorporated preference information in an achievement scalarization function. Zitzler et al. [21] proposed a general multiobjective optimizer that can be adapted to arbitrary user preferences assuming that the goal is to approximate the Pareto-optimal set. Molina et al. [15] proposed a variation of the concept of Pareto dominance, called g-dominance, which is based on the information included in a reference point and designed to be used with any multiobjective metaheuristics.

In this work, we use the concept of reference point in a tabu/scatter search approach and we are interested in achieving an accurate approximation to the set of preferred Pareto solutions.

The remainder of the paper is organized as follows. In Section 2, basic concepts are presented. Section 3 describes the multiobjective scatter search based on reference point. Section 4 gives the computational experiments. Section 5 contains results and discussion. Conclusion is presented in Section 6.

## 2 Basic concepts

A general multiple objective optimization (MOO) problem consists of optimizing a set of  $r \geq 2$  objective functions. It can be formulated as follows:

$$\text{minimize}\{f(x) : f(x) = (f_1(x), f_2(x), \dots, f_r(x))\}$$

s.t.

$$x \in X$$

where a solution  $x = (x_1, x_2, \dots, x_n) \in X$  is represented by a vector of  $n$  decision variables,  $X$  is a set of feasible solutions.

The image of a solution  $x$  in the objective space is a point

$$z = (z_1, z_2, \dots, z_r) = f(x).$$

Having several objective functions, the notion of optimum changes. The aim here is to find a good compromise rather than a unique solution as in a single-objective optimization problem. A MOO problem obtains a set of solutions known as the Pareto optimal, related to the following concepts.

**Definition 1 (Pareto Dominance)** A solution  $x^1 \in X$  dominates another solution  $x^2 \in X$  if and only if  $\forall i \in \{1, 2, \dots, n\}$ ,  $f_i(x^1) \leq f_i(x^2)$ , and  $\exists j \in \{1, 2, \dots, n\} : f_j(x^1) < f_j(x^2)$ .

**Definition 2 (Efficiency)** A solution  $x^*$  is efficient if and only if there is not another solution  $x \in X$  such that  $x$  dominates  $x^*$ .

The whole set of efficient solutions is the Pareto optimal set, and is denoted by  $X_P$ . The image of a Pareto optimal set in the objective space results in a set of non-dominated vectors denoted by  $PF$  and called *nondominated set* or Pareto frontier.

The aim in multiobjective metaheuristic optimization is to obtain a Pareto optimal set or a good approximation to it. This is a very difficult task and it depends on the practical complexity of the problem. As we said above, the introduction of preference information permits us to narrow the search over the regions of interest of the decision-maker.

## 2.1 Preference information and scalarizing functions

The scalarizing functions approaches, proposed by Wierzbicki ([18],[19]) where the preference information given by a reference point does not need to be an attainable point. These functions can be stated as follows:

$$s(z, \bar{z}, \xi) = \min\{\xi \min_j (z_j - \bar{z}_j), \sum_{j=1}^r (z_j - \bar{z}_j)\} + \sum_{j=1}^r (z_j - \bar{z}_j) \quad (1)$$

where  $\xi$  is a parameter not less than the dimension of the objectives  $r$ .

$$s(z, \bar{z}, \rho, ) = \min\{\max_j \{(z_j - \bar{z}_j)\} + \rho \sum_{j=1}^r (z_j - \bar{z}_j)\} \quad (2)$$

where  $\rho$  is an arbitrary small positive parameter ( $0 < \rho \leq 1$ ). The point  $\bar{z}_j$  denote the reference point.

The expression (2) is used to design our selection method. The expression (1) and other scalarizing functions can be used to develop a parallel approach.

### 3 A brief about the MOSS-II algorithm

Here we present the principal components of the MOSS-II algorithm [3].

#### 3.0.1 Neighborhood

We use a simplified version of a sequential fan strategy as a candidate list strategy. The sequential fan generates  $p$  best alternative moves at a given step, and then creates a fan of solution streams, one for each alternative. The best available moves for each stream are again examined, and only the  $p$  best moves overall provide the  $p$  new streams at the next step. In our case, taking  $p = 1$ , we have in each step one stream and a fan of 60 points to consider. We propose to select moves that consist of changing at most five variables. The range of each variables is split into subranges, frequency memory is used to control the selection of the subranges where the variables take values.

Tabu restrictions are imposed to prevent moves that bring the values of variables “too close” to values they held previously.

#### 3.0.2 Transitions

Now we explain how to transit to a new solution. Let  $E$  the set of efficient moves and  $D$  the set of deficient moves, where a deficient move is a move that not satisfies the aspiration level, in otherwise the move is efficient.

**Definition 3** *The best efficient move denoted by  $m^*$ , is defined as a move such that  $[m^* \in E(x) : f(m^*(x)) = \min\{f(x') : x' = m(x); \forall m \in E(x)\}]$ .*

**Definition 4** *The best move denoted by  $m_{best}$  is defined as a move such that  $m_{best}$  is equal to  $m^*$  if  $E(x) \neq \emptyset$ , otherwise is equal to  $m'$  such that  $m' \in D(x) : f(m'(x)) = \min\{f(m(x)); \forall m \in D(x)\}$ .*

#### 3.0.3 Aspiration level

A thresholding aspiration is used to obtain an initial set of solutions as follows: without lost generality, assume that every criteria is minimized. Let  $\Delta f(x') = (\Delta f_1(x'), \dots, \Delta f_r(x'))$  where  $\Delta f_i(x') = f_i(x') - z_i^*$ ,  $i \in \{1, \dots, r\}$  and  $Z^*$  is a reference solution,  $Z^* = (z_1^*, \dots, z_r^*)$ . Then,  $x$  is accepted to introduce into  $S$  if  $(\exists \Delta f_i(x') \leq 0)$  or  $(\forall i \in \{1, \dots, r\} [\Delta f_i(x') = 0])$ , otherwise is rejected. The point  $Z^*$  is updated by  $z_i^* = \min f_i(x') \forall i \in \{1, \dots, r\}, x' \in S$ .

An additive function value denoted by  $afv$  with weights  $\lambda_i = 2 - \exp(-s_i)$  is taken to measure the quality of the solution, where  $s_i = \left| \frac{f_i(x') - z_i^*}{|z_i^*|} \right|$ . if  $z_i^* = 0$  then we take only the absolute value of  $\Delta f_i(x')$ .

Let

$$afv(x') = \{(1 - \theta) * \sum_{i=1,r} \lambda_i \Delta f_i(x')\}.$$

The parameter  $\theta$ , is a parameter associated to the selected variables and the selected subranges, that takes the value 0 if the number of time that the subrange  $j$  has been visited by the variable with index  $i$  is greater than a certain threshold.

### 3.1 Duplicated points

As in all our MOSS algorithms avoiding the duplicate points already generated is a significant factor in producing an effective overall procedure. The generation of a duplicate point is called a critical event. Our algorithm is based on a “critical event design” that monitors the current solutions in the reference set  $R$ , containing the current efficient points, and in the trial solutions set  $S$ . The elements considered in the critical event design are the values of the objectives, and the decision variables. We consider that a critical event takes place if one trial solution is “too close” to another solution belonging to the trial solution set or to the reference set.

### 3.2 Choosing diverse subsets of nondominated points

As a basis for creating combined solutions we generate subsets. Our approach is organized to generate three different collections of diverse subsets, which we refer to as  $D1$ ,  $D2$ , and  $D3$  subsets of  $R$ .

Suppose  $R1 \neq \emptyset$  and  $R2 \neq \emptyset$ ,  $R1, R2 \subseteq R$ ,  $T_D$  a set of forbidden solutions, composing a subset of  $R$  excluded from consideration to be combined during  $t$  scatter iterations, and  $T1_D$  a set of forbidden solutions, composing a subset of  $R$ , excluded from consideration to be combined during  $t1$  scatter iterations,  $D \in \{D1, D2, D3\}$ . Then, the type of subsets we consider are as follows:

- 3-element subsets  $D1$ , where the first element is in  $R1 - T_{D1}$ , the second element pertains to  $R1 - T1_{D1}$  and it is the most dissimilar to the first, and the third element belongs to  $R1 - T1_{D1}$  selected to be the most dissimilar to the former two.
- 3-element subsets  $D2$ , where the first element is in  $R1 - T_{D2}$ , the second element pertains to  $R2 - T1_{D2}$  and it is the most dissimilar to the first, and

the third element belongs to  $R2 - T1_{D2}$  selected to be the most dissimilar to the former two.

- 3-element subsets  $D3$ , where the first element is in  $R2 - T_{D3}$ , the second element pertains to  $R1 - T1_{D3}$  and it is the most dissimilar to the first, and a third element that belongs to  $R1 - T1_{D3}$  selected to be the most dissimilar to the selected elements.

If  $R1 - Z$  or  $R2 - Z$  empty for  $Z = T_{D \in \{D1, D2, D3\}}$  or  $Z = T1_{D \in \{D1, D2, D3\}}$  then, we take a random solution of  $R1$  or  $R2$  respectively.

### 3.3 Linear combinations

Our strategy consist to create  $\Omega(D) = x + w(y - x)$ , for  $w = 1/2, 1/3, 2/3, 3/4, 4/5, 9/10, -1/3, -2/3, 4/3, 5/3$ , and  $x, y \in D$  and to generate new trial points on lines between  $(x$  and  $y)$ ,  $(x$  and  $z)$ ,  $(y$  and  $z)$ ,  $(x'$  and  $z)$ ,  $(y'$  and  $y)$ ,  $(z'$  and  $x)$ ,  $(c$  and  $x')$ ,  $(c$  and  $y')$ ,  $(c$  and  $z')$ , where  $(y' = y + \frac{x-z}{2})$ ,  $(z' = z + \frac{y-z}{2})$ ,  $(x' = x + \frac{x-y}{2})$  and  $(c = (x + y + z)/3)$ , here  $x'$   $y'$   $z'$  corresponds to  $x_{trial}$   $y_{trial}$   $z_{trial}$  respectively. For more details see [4].

## 4 Multiobjective tabu/scatter search based on reference point (RP-MOSS)

RP-MOSS is a tabu/scatter search method based on reference point approach. The main idea of our work is to incorporate preference information into the search and to propose an interactive multiobjective tabu/scatter search approach. Preference information is given in terms of reference point. Reference points consist of aspiration levels reflecting desirable values for the objective functions. This is a natural way of expressing preference information and in this straightforward way the decision-maker (DM) can express wishes about improved solutions and directly see and compare how well they could be attained when the next solutions are generated. The information is used to concentrate the search on certain portions of the Pareto front. A choice method that uses a scalarizing function is used to choose a subset of nondominated points near the preferred region to guide the search. A new strategy for generating subsets of solutions of the reference set is introduced. The method attempts to find a good approximation of Pareto-optimal solutions satisfying the supplied goal in only one run.

General characteristics of our approach are summarized as follows:

1. A deterministic approximation approach.
2. Reference-based strategy embedded in our tabu/scatter search.
3. The reference point may be an attainable or a unattainable point.
4. The method is indifferent to the shape of the Pareto-optimal frontier (such as convex or non-convex, continuous or discrete, connected or disconnected and others).
5. The method is indifferent to the geometrical shapes of the Pareto set.
6. The DM can change the preference information, new reference points are given in a progressive interaction to evaluate other projections.
7. The method is applicable to a large number of objectives (say, 10 or more), a large number of variables and linear or non-linear constraints.

The following updating to our previous approaches is performed:

1. The Kramer Choice Function is changed by a new selection method that uses an scalarizing function of normalized values.
2. A modified strategy to chose subsets of non-dominated points to be combined is introduced.
3. Frequency Memory is used to select the variables and the sub-ranges, choosing the most frequent variable and its associated sub-range in the intensification strategy, and the less frequently in the diversification strategy.
4. The bounds of the sub-ranges are changed in each interactive iteration.

#### 4.1 Choice method

Without loss of generality we assume that all attributes are minimized.

1. Normalize the value of the elements of  $R$ , let

$$\tilde{R} = \{\tilde{z} | \tilde{z} = |(z - \bar{z})/z^*|\}$$

where  $z^*$  is the maximum value for the solutions  $z \in R$  (we assume that  $z^* \neq 0$ ) and  $\bar{z}$  is the reference point.

## 2. Calculate the Choice Function

$$M_{\tilde{R}}^* = \min_{\tilde{z} \in \tilde{R}} M_{\tilde{R}}(\tilde{z})$$

$$\text{where } M_{\tilde{R}}(\tilde{z}) = \max_j \{\tilde{z}_j\} + \rho \sum_{j=1}^r \tilde{z}_j$$

$$C(\tilde{R}) = \{\tilde{z} \in \tilde{R} | M_{\tilde{R}}^* \leq M_{\tilde{R}}(\tilde{z}) \leq (4/3)M_{\tilde{R}}^*\}$$

we take  $\rho = 0.05$ .

## 3. Set $R = C(\tilde{R})$ the reference set to be combined.

We try to maintain fifteen preferred solutions in each iteration. The idea is to avoid overloading the DM with too much information. To do this, the set  $C(\tilde{R})$  is sorted in ascending order, then it is updated with a number of solutions equal to the  $\min(|C(\tilde{R})|, 15)$  taken in the corresponding order.

The function  $M_{\tilde{R}}$  is a variant of the scalarizing function (2).

### 4.2 Generating subsets method

As a basis for creating combined solutions we generate subsets  $D \subset R$ .

The type of subset we consider is as follows: 2-element subsets  $D$  where the first element is in  $R - T_D$  and the second pertains to  $R - T_D$  selected to be the closest to the first (the minimum Euclidean distance).

$T_D$  is a forbidden set of solutions that were selected to be combined.

### 4.3 Linear combinations method

Our strategy consists on creating  $\Omega(D) = x + (y - x)$ , for  $= 1/3, 1/2, 2/3, 3/4, 4/5$ , and  $x, y \in D$  as follows:

Generate new trial points on lines between  $x$  and  $y$ .

### 4.4 Changing the bounds of the sub-range

For the current selected set of non-dominated solutions is calculated the maximum and the minimum value of each variable. Then, for each variable defined in the range  $[l_i, u_i]$  (as in our previous approaches MOSS [3] and MOSS-II, each range is divided in sub-ranges) the lower bound of the first sub-range is set equal to  $l_i$ , and the upper bound of the last sub-range is set equal to  $u_i$ . Next,

the upper bound of the first sub-range is set equal to the minimum value of the corresponding variable, and the lower bound of the last sub-range is equal to the maximum of the corresponding variable also. A number of equal-sized sub-ranges are formed between these two boundaries.

This mechanism permits to intensify the search in the region defined by the current non-dominated solutions permit us to restrict the search to this region. Nevertheless, the bound sub-ranges (the first and the last sub-ranges) permit to escape of the narrow region.

#### 4.5 RP-MOSS procedure

The basic idea of our new version is the progressive and select search of potentially non-dominated (p.n.d.) solutions by focusing the search, in each step of interaction, in a sub-region of point closest to the reference point supplied by the decision maker. Each computing phase produces a set of p.n.d. solutions more concentrated around the best p.n.d. solution, taking into account the choice function previously described. As in our previous approaches, we use a multi-start tabu search as a generator of diverse solutions. Initially, a set of trial solutions are created from the starting points and these are improved by linear combination. These solutions are filtered to choose feasible trial solutions. When all starting solutions have been explored, the parameter  $\delta$ , used to avoid duplicated points, is reduced using the following expression  $\delta = (0.8)\delta$ . The obtained solutions are improved in the scatter phase by the linear combination method. These solutions are presented to the DM, and he/she can decide to stop or to continue the search. He can continue the search changing the reference point or not. The new reference point is calculated as a convex combination between the current reference point and the new reference point supplied by the DM (i.e. update  $\bar{z}$  using  $\bar{z} = (1 - \theta)\bar{z} + \theta\hat{z}$  where  $\hat{z}$  is the new reference point, and the user-parameter satisfying  $0 \leq \theta \leq 1$ ). To continue, by re-starting the TS approach, new non-dominated solutions are generated from a subset of potentially Pareto solutions nearest to the reference point supplied by the DM.

A formal algorithm of RP-MOSS is given below.

#### Algorithm RP-MOSS

```

Create starting points to initiate the approach
Repeat
  While (iter  $\neq$  2)//tabu phase
    Repeat
      Use of a memory-based strategy to generate
      a set of trial solutions from the starting points

```

Filter to preserve the feasibility  
 Update the set of reference points taking  
 the current potentially Pareto solutions  
 Until to explore all starting points  
 Modify the parameter  $\delta$   
 Choose new starting points from the reference set  
 End //the tabu phase  
 //begin the scatter phase  
 Generate subsets of the reference set using  
 the Generating Subset Method  
 Apply the Linear Combination Method to obtain new solutions  
 Update the reference set using the Choice Method  
 Update the starting points from the reference set  
 //end the scatter phase  
 Change the sub-ranges  
 Show the solutions obtained to the decision-maker  
 Until (the decision-maker is satisfied)

## 5 Numerical examples

The proposed approach is tested on a set of multiobjective test problems from two to 10 objectives and high dimensionality.

### Two-objective test problems

First, we consider the 100-variable modified ZDT1 and modified ZDT2 problems [16]. These problems have a convex Pareto-optimal front and a concave Pareto-optimal front respectively, spanning continuously in  $f_1 \in [0, 1]$ .

$$\text{Minimize } ZDT(x) = (f_1, f_2)$$

subject to

$$f_2 = g(X)(1 - \sqrt{x_1/g(X)})$$

$$x = (x_1, \dots, x_n)$$

where  $g(x) = 1 + 9/(n-1) \sum_{i=2}^n x_i^2$ .

$x_1 \in [0, 1]$  and  $x_i \in [-1, 1]$  for  $i \in \{2, 3, \dots, n\}$

The Pareto-optimal solutions correspond to  $0 \leq x_1 \leq 1$  and  $x_i = 0$  for  $i \in \{2, 3, \dots, n\}$ .

For the modified ZDT2 the function  $f_2$  takes the following form:

$$f_2 = g(X)(1 - (x_1/g(X))^2).$$

### Three-objective test problem

Problem LZ07\_F6 [13]: This problem has a concave PF and there are non-linear linkages between decision variables.

$$\text{Minimize } f_1 = \cos(0.5x_1\pi) \cos(0.5x_2\pi) + \frac{2}{|J_1|} \sum_{j \in J_1} (x_j - 2x_2 \sin(2\pi x_1 + \frac{j\pi}{n}))^2$$

$$\text{Minimize } f_2 = \cos(0.5x_1\pi) \sin(0.5x_2\pi) + \frac{2}{|J_2|} \sum_{j \in J_2} (x_j - 2x_2 \sin(2\pi x_1 + \frac{j\pi}{n}))^2$$

$$\text{Minimize } f_3 = \sin(0.5x_1\pi) + \frac{2}{|J_3|} \sum_{j \in J_3} (x_j - 2x_2 \sin(2\pi x_1 + \frac{j\pi}{n}))^2$$

where  $J_1 = \{j | 3 \leq j \leq n \text{ and } j - 1 \text{ is a multiplication of } 3\}$

$J_2 = \{j | 3 \leq j \leq n, \text{ and } j - 2 \text{ is a multiplication of } 3\}$

$J_3 = \{j | 3 \leq j \leq n, \text{ and } j \text{ is a multiplication of } 3\}$ ;

Pareto set =  $\{x | x_j = 2x_2 \sin(2\pi x_1 - \frac{j\pi}{n}), j = 3, \dots, n\}$ .

Pareto front =  $f_1^2 + f_2^2 + \dots + f_r^2 = 1$  with  $f_1, f_2, f_3 \in [0, 1]$

Here we use  $n = 10$  recommended by the authors.

### Five-objective test problems

Miettinen et al. [14]: The problem is related to locating a pollution monitoring station in a two-dimensional decision space. This problem is highly non-linear and the Pareto optimal set is discontinuous.

$$f_1(x) = -u_1(x_1, x_2) - u_2(x_1, x_2) - u_3(x_1, x_2) + 10$$

$$f_2(x) = -u_1(x_1 - 1.2, x_2 - 1.5) - u_2(x_1 - 1.2, x_2 - 1.5) - u_3(x_1 - 1.2, x_2 - 1.5) + 10$$

$$f_3(x) = -u_1(x_1 + 0.3, x_2 - 3.0) - u_2(x_1 + 0.3, x_2 - 3.0) - u_3(x_1 + 0.3, x_2 - 3.0) + 10$$

$$f_4(x) = -u_1(x_1 - 1.0, x_2 + 0.5) - u_2(x_1 - 1.0, x_2 + 0.5) - u_3(x_1 - 1.0, x_2 + 0.5) + 10$$

$$f_5(x) = -u_1(x_1 - 0.5, x_2 - 1.7) - u_2(x_1 - 0.5, x_2 - 1.7) - u_3(x_1 - 0.5, x_2 - 1.7) + 10$$

$$u_1(x) = 3(1 - x_1)^2 \exp(-x_1^2 - (x_2 + 1)^2)$$

$$\begin{aligned}
 u_2(x) &= -10(x_1/4 - x_1^3 - x_2^5) \exp(-x_1^2 - x_2^2) \\
 u_3(x) &= 1/3 \exp(-(x_1 + 1)^2 - x_2^2) \\
 x_1 &\in [-4.9, 3.2], \quad x_2 \in [-3.5, 6].
 \end{aligned}$$

### 10-objective test problem

Problem DTLZ2 [10]: The following test problem prove the ability of our approach to project the reference point on the global Pareto-optimal front, satisfying  $f_1^2 + f_2^2 + \dots + f_r^2 = 1$  in the range  $f_1, f_2, \dots, f_r \in [0, 1]$ , we considered 19 variables and 10 objectives.

### CTP2 constrained test problem

This test problem was proposed by Deb in [9]. In it the Pareto-optimal region is a disconnected set of the unconstrained Pareto-optimal feasible region.

$$\begin{aligned}
 &\text{minimize } f_1(x) = x_1, \\
 &\text{minimize } f_2(x) = g(x)(1 - f_1(x)/g(x)). \\
 &\text{s.t.} \\
 &c(x) = \cos(\theta)(f_2(x) - e) - \sin(\theta)f_1(x) \geq a|\sin(b\pi(\sin(\theta)(f_2(x) - e) + \\
 &\cos(\theta)f_1(x))^c)|^d \\
 &x_1 \in [0, 1], \quad -5 \leq x_i \leq 5, \quad (i = 2, 3, 4, 5).
 \end{aligned}$$

The following parameters are used to define the problems:

$$\theta = -0.2\pi, \quad a = 0.2, \quad b = 10, \quad c = 1, \quad d = 0.5, \quad e = 1.$$

For the above problems, the Pareto-optimal solutions lie on the straight line  $(f_2(x) - e) \cos(\theta) = f_1(x) \sin(\theta)$ .

### Welded beam design constrained real test problem

The welded beam design problem has four real-parameter variables  $x = (h(x_1), l(x_2), t(x_3), b(x_4))$  and four non-linear constraints. One of the two objectives is to minimize the cost of fabrication and other is to minimize the end deflection of the welded, the task is to find, if possible, a set of solutions which are better than the given reference point in all objectives (see [8]).

$$\text{Minimize } f_1(x) = 1.1047x_1^2x_2 + 0.04811x_3x_4(14.0 + x_2)$$

$$\text{Minimize } f_2(x) = \frac{2.1952}{x_3^3}$$

s.t.

$$g_1(x) = 13,600 - \tau(x) \geq 0,$$

$$g_2(x) = 30,000 - \sigma(x) \geq 0,$$

$$g_3(x) = x_4 - x_1 \geq 0,$$

$$g_4(x) = 0.10471(x_1^2) - 0.04811x_3x_4(14.0 + x_2) + 5.0 \geq 0,$$

$$g_5(x) = x_1 - 0.125 \geq 0,$$

$$0.125 \leq x_1, x_2 \leq 5$$

$$0.1 \leq x_3, x_4 \leq 10.$$

The terms  $\tau(x)$ ,  $\sigma(x)$ ,  $P_c(x)$ ,  $\delta(x)$  are given below

$$\tau(x) = \sqrt{(\tau')^2 + (2\tau'\tau'')\frac{x_2}{2R} + (\tau'')^2}$$

$$\tau'(x) = \frac{6000}{\sqrt{2}x_1x_2}$$

$$\tau''(x) = \frac{6000(14 + \frac{x_2}{2})\sqrt{0.25(x_2^2) + ((x_1 + x_3)/2)^2}}{2[x_1x_2\sqrt{2}(x_2^2/12 + 0.25(x_1 + x_3)^2)]}$$

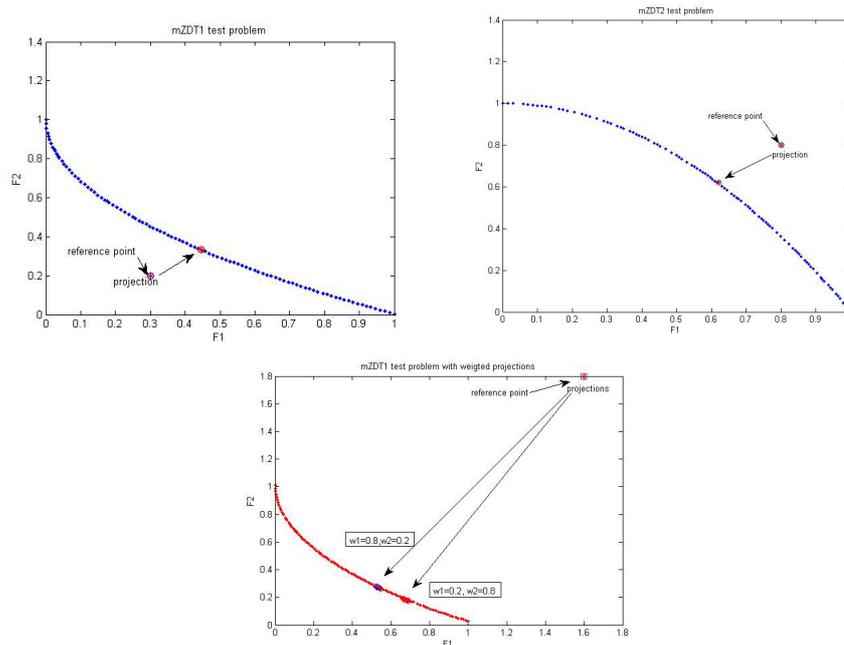
$$\sigma(x) = \frac{504,000}{x_3^2x_4}$$

$$\delta(x) = \frac{65,856,000}{(30 \times 10^6)x_4x_3^3}$$

$$P_c(x) = \frac{4.013(30 \times 10^6)\sqrt{\frac{x_3^2x_4^6}{36}}}{196} \left( 1 - \frac{x_3\sqrt{\frac{30 \times 10^6}{4(12 \times 10^6)}}}{28} \right).$$

## 6 Results and discussion

The following figures illustrate the proposed interactive approach by which a set of Pareto-optimal solutions or a good approximation to it near a supplied reference point will be found. For a chosen reference point, the closest set of Pareto-optimal solutions is the target solution to the reference point method. If the decision-maker is not satisfied with the achieved solution, her/his can continue with the same reference point or can change the reference point and to continue the search.



**Figure 1:** Preferred solutions on the mZDT1 and mZDT2 test problems.

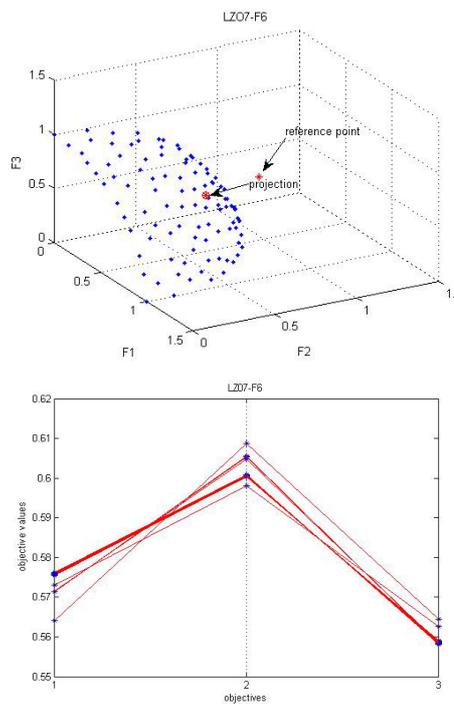
Figure 1 shows the approximations for the mZDT1 and mZDT2 test problems. In the upper pictures at left, a unattainable (infeasible) point (0.3,0.2) is given, and at right, is given an attainable (feasible) reference point (0.8,0.8). Observe that the obtained solutions are non-dominated points that lies practically over the Pareto front and close to the supplied reference point. You can see that the non-convexity does not cause difficulty to the proposed approach.

In the lower picture, two projections of the same reference point (1.6,1.8) are shown. One of these projections is obtained with weights (0.8,0.2), you can

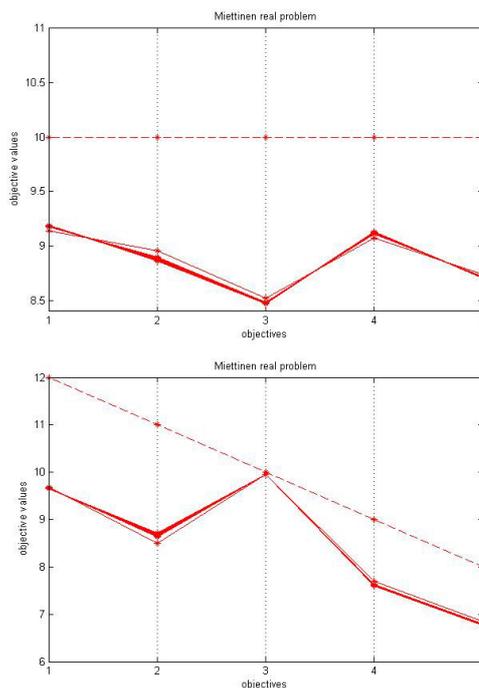
observe, that in this case the objective  $f_1$  is preferred. In the second projection, the weights are (0.2,0.8) given more preference to the objective  $f_2$ .

In these cases to measure how far the nondominated solutions achieved are from the Pareto optimal front, the measure Generational Distance suggested in [20] is used,  $GD = (\sum_{i=1}^{|Q|} d_i^p)^{\frac{1}{p}} / |Q|$  where  $p = 2$ . For the test problem mZDT2 and the reference point (0.8,0.8), the GD value is 0. For the mZDT1 test problem, reference point (0.3,0.2), the GD value is 0.0006. This means, that in the first case the projection lies on the optimal-Pareto front, and in the second case very close to it. The points of the projection for the reference point (0.3,0.2) lie in the following ranges:  $f_1 \in [0.445407, 0.445413]$  and  $f_2 \in [0.335165, 0.335174]$  and for the reference point (0.8,0.8)  $f_1 \in [0.621428, 0.621945]$  and  $f_2 \in [0.620823, 0.621283]$ .

Figure 2. shows how our approach is also indifferent to the geometrical shape of the Pareto set. In this picture the reference point is (0.8,0.8,0.8), and the projections are close to the surface of the Pareto optimal front.  $\sum_{i=1}^3 f_i^2 \in [1.007, 1.007]$  (for an approximate 0.7% outside from one) that means that the projection is very close to the optimal-Pareto front.



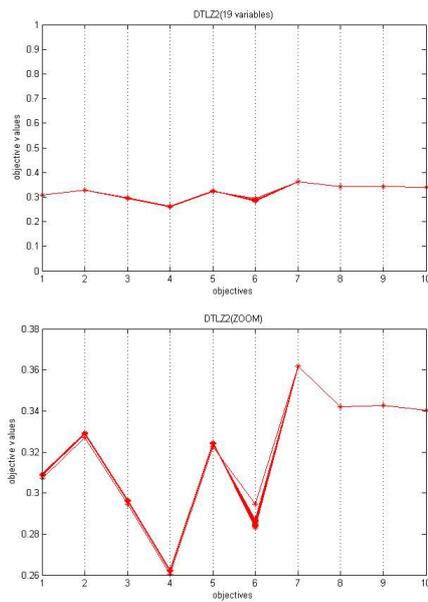
**Figure 2:** Preferred solutions on the LZ07-F6 test problem.



**Figure 3:** Preferred solutions on the Miettinen test problem.

Figure 3. shows the approximation obtained for the Miettinen real test problem with reference points (10, 10, 10, 10, 10) on the left, and (12, 11, 10, 9, 8) on the right. The reference points are indicated by straight lines. Observe that the projections lie under the reference points. An appropriate number of preferred solutions and a graphical representation, give the possibility to make a comparison and to make a preferred decision.

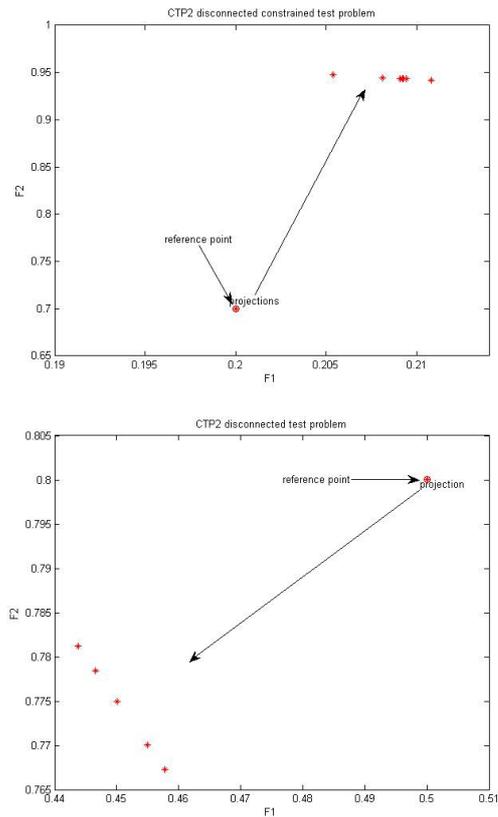
Figure 4. shows the preferred solutions achieved on the DTLZ2 for 10-objectives, 19 variables. The figure shows the graphic for the DTLZ2, the reference point is (0.3,0.3,...,0.3). In this case the solutions are also around the supplied reference point and near to the Pareto-optimal front  $\sum_{i=1}^{10} f_i^2 \in [1.027, 1.027]$  (for an approximate 2.7% outside from one). A zoom of the reached solutions shows that the solutions range between [0.26, 0.37] indicating that these are around the reference point given by the DM. This indicate the ability of this approach to solve difficult problems with a great number of objectives.



**Figure 4:** Preferred solutions on the DTLZ2 test problem.

Now we present a constrained disconnected test problem CTP2 figure 5. In the left side is given the unattainable reference point  $(0.2, 0.7)$ , and in the right side the attainable reference point  $(0.5, 0.8)$ . The figure 5 shows that the supplied reference points are not optimal solutions and there exist a number of solutions which dominate these solutions. The aim is to obtain non-dominated solutions over dominated solutions. In these two cases the approach achieved a set of disconnected solutions very close to the optimal Pareto front, with a deviation of 0.07 and 0.08 respectively from the straight line  $(f_2(x) - e) \cos(\theta) = f_1(x) \sin(\theta)$ .

Figure 6 illustrates the capacity of our approach to help the decision-maker (DM) in the case that his/her is interested in knowing a tradeoff in different regions of interest. Different feasible and unfeasible reference points are given in an interactive approach and different projections near to they are presented to the DM. To investigate where these regions are with respect to the complete trade-off front, we also show the original MOSS-II solutions. Here a first reference point in  $(5, 0.003)$  is supplied. The supplied point is not a feasible solution and the proposed approach obtains feasible solutions improving, in the Pareto sense, the solutions achieved by MOSS-II. From here, the DM continues by varying the



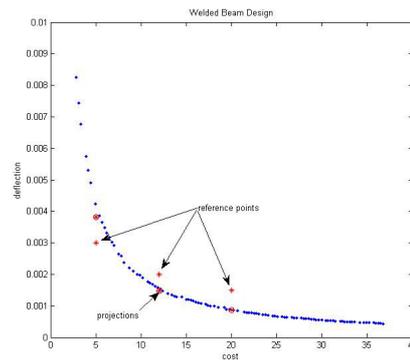
**Figure 5:** Preferred solutions on the CTP2 test problem.

reference point by supplying two new feasible targets (12,0.002) and (20,0.0015) the news projections fall in the frontier.

Thus, if the decision-maker is interested in knowing a trade-off of optimal solutions in different regions, the proposed interactive approach is able to achieves non-dominated solutions near to the supplied targets.

## 7 Conclusion

In this paper, we introduce a new preference-based tabu/scatter search approach that incorporate preference information coming of the decision-maker. Our main purpose is, to develop a deterministic tabu/scatter search to find a set of solutions in the region of Pareto-optimality, which are of interest to the decision-maker. We show that, only a few changes, principally the selection method, are



**Figure 6:** Preferred solutions on the Welded Beam real test problem.

necessary to incorporate preference information in the tabu/scatter search architecture. An adaptive parameter control adjusts the neighborhood of the projected reference point to show a reasonable number of solutions to the DM. We also show, how changing the reference point in an interactive approach the decision-maker can explore different preference regions of the Pareto-optimality.

The proposed technique has been applied to a number of 2-objective to 10-objective optimization problems, from 2 to 100 variables, and with different shapes and restrictions. In all cases, a set of nondominated preferred solutions has been obtained.

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